JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **94**, 38–43 (1983)

Entropy of a Finite Partition of Fuzzy Sets

KEN KURIYAMA

Department of Applied Mathematics,
Faculty of Engineering,
Yamaguchi University, Ube, Japan

Submitted by L. Zadeh

1. INTRODUCTION

In information theory the concept of entropy of an information source is important. In ergodic theory as well as in information theory the entropies of measure preserving transformations are important because they are invariant under isomorphisms. In order to define the entropy of a measure preserving transformation we must define the entropy and the conditional entropy of a finite partition of measurable subsets.

The purpose of this paper is to define the entropy of a finite partition of measurable fuzzy sets and show the properties. The entropies of finite partitions of fuzzy sets are important for applications to theory of fuzzy informations and fuzzy messages.

2. ENTROPY AND CONDITIONAL ENTROPY

Let $(X, \mathfrak{F}, P)$ be a probability measure space. For a fuzzy set A of X its membership function is denoted by m_A. In particular, if $m_A(x) = 0$ for all $x \in X$, A is the empty set $\varnothing$. For fuzzy sets A and B of X the product AB means that $m_{AB}(x) = m_A(x)\, m_B(x)$. The inclusion relation $A \subset B$ means that $m_A(x) \leqslant m_B(x)$ for $x \in X$. A fuzzy set A of X is said to be measurable if the membership function m_A is measurable. The measure $P(A)$ of measurable fuzzy set A is defined by $P(A) = \int_X m_A(x)\, dP(x)$.

DEFINITION 1. A family $\xi = \{A_1, \ldots, A_l\}$ of measurable fuzzy sets of X is called a finite partition of fuzzy sets if

$$\sum_{i=1}^{l} m_{A_i}(x) = 1 \qquad \text{for} \quad x \in X$$

and $A_i \neq \varnothing$ for all $1 \leqslant i \leqslant l$.

0022-247X/83 $3.00

It is obvious that $\sum_{i=1}^{l} P(A_i B) = P(B)$ if $\{A_1, ..., A_l\}$ is a finite partition of fuzzy sets and B is a fuzzy set.

DEFINITION 2. Let $\xi = \{A_1, ..., A_l\}$ and $\eta = \{B_1, ..., B_m\}$ be finite partitions of fuzzy sets. We set $\xi\eta = \{A_i B_j : A_i B_j \neq \varnothing\}_{i,j}$. The family $\xi\eta$ of fuzzy sets is called the product of ξ and η.

PROPOSITION 3. *The product of two finite partitions of fuzzy sets is a finite partition of fuzzy sets.*

Proof. Let $\xi = \{A_1, ..., A_l\}$ and $\eta = \{B_1, ..., B_m\}$ be finite partitions of fuzzy sets. We put $K = \{(i,j) : 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant m,\ A_i B_j \neq \varnothing\}$ and for each i we set $J(i) = \{j : (i,j) \in K\}$.

Let $x_0 \in X$ fix. If $m_{A_i}(x_0) \neq 0$, we get

$$\begin{aligned}\sum_{j \in J(i)} m_{A_i B_j}(x_0) &= \sum_{j \in J(i)} m_{A_i}(x_0)\, m_{B_j}(x_0) \\ &= m_{A_i}(x_0) \sum_{j \in J(i)} m_{B_j}(x_0) = m_{A_i}(x_0).\end{aligned}$$

If $m_{A_i}(x_0) = 0$, it is obvious that

$$\sum_{j \in J(i)} m_{A_i B_j}(x_0) = m_{A_i}(x_0).$$

Hence

$$\begin{aligned}\sum_{(i,j) \in K} m_{A_i B_j}(x_0) &= \sum_i \sum_{j \in J(i)} m_{A_i}(x_0)\, m_{B_j}(x_0) \\ &= \sum_i m_{A_i}(x_0) = 1.\end{aligned}$$

Therefore $\xi\eta$ is a finite partition.

DEFINITION 4. Let $\xi = \{A_1, ..., A_l\}$ and $\eta = \{B_1, ..., B_m\}$ be finite partitions of fuzzy sets. The finite partition η is said to be a refinement of ξ if for each $A_i \in \xi$ there exists a subset $J(i)$ of $\{1, ..., m\}$ such that $m_{A_i} = \sum_{j \in J(i)} m_{B_j}$, $J(i) \cap J(h) = \varnothing$ for $i \neq h$ and $\bigcup_{i=1}^{l} J(i) = \{1, ..., m\}$.

It is denoted by $\xi < \eta$ that η is a refinement of ξ.

Remark 5. (i) If $\eta = \{B_1, ..., B_m\}$ is a refinement of $\xi = \{A_1, ..., A_l\}$, $P(A_i) = \sum_{j \in J(i)} P(B_j)$.

(ii) The product $\xi\eta$ of finite partions ξ and η is a refinement of ξ and η.

(iii) The relation $\xi < \eta$ does not necessarily imply $\xi\eta = \eta$.

Now we define the entropy of a finite partition of fuzzy sets.

DEFINITION 6. Let $\xi = \{A_1, ..., A_l\}$ be a finite partition of fuzzy sets. The entropy $H(\xi)$ of ξ is defined as follows:

$$H(\xi) = -\sum_{i=1}^{l} P(A_i) \log P(A_i) \qquad \text{where} \quad 0 \times \log 0 = 0.$$

DEFINITION 7. Let $\xi = \{A_1, ..., A_l\}$ and $\eta = \{B_1, ..., B_m\}$ be finite partitions of fuzzy sets. The conditional entropy $H(\xi | \eta)$ of ξ given η is defined as follows:

$$H(\xi | \eta) = -\sum_j P(B_j) \sum_i \frac{P(A_i B_j)}{P(B_j)} \log \frac{P(A_i B_j)}{P(B_j)}$$
$$= -\sum_{i,j} P(A_i B_j) \log \frac{P(A_i B_j)}{P(B_j)},$$

where j terms are omitted when $P(B_j) = 0$.

Remark 8. We easily get the following:

(i) $H(\xi) \geqslant 0$ and $H(\xi | \eta) \geqslant 0$.

(ii) When $\eta = \{X\}$, $H(\xi | \eta) = H(\xi)$.

Now we can generalize [3, Theorem 4.3] to the case of finite partitions of fuzzy sets. Theorem 9 can be proved almost similarly to [3, Theorem 4.3]; however, we shall prove it for the convenience of the reader.

THEOREM 9. *Let ξ, η, and ζ be finite partitions of fuzzy sets. Then*

(i) $H(\xi\eta | \zeta) = H(\xi | \zeta) + H(\eta | \xi\zeta)$,

(ii) $H(\xi\eta) = H(\xi) + H(\eta | \xi)$,

(iii) if $\xi < \eta$, $H(\xi | \zeta) \leqslant H(\eta | \zeta)$,

(iv) if $\xi < \eta$, $H(\xi) \leqslant H(\eta)$,

(v) if $\eta < \zeta$, $H(\xi | \eta) \geqslant H(\xi | \zeta)$,

(vi) $H(\xi | \zeta) \leqslant H(\xi)$,

(vii) $H(\xi\eta | \zeta) \leqslant H(\xi | \zeta) + H(\eta | \zeta)$, and

(viii) $H(\xi\eta) \leqslant H(\xi) + H(\eta)$.

Proof. We set $\xi = \{A_1, ..., A_l\}$, $\eta = \{B_1, ..., B_m\}$, and $\zeta = \{C_1, ..., C_n\}$. Without loss of generality we may assume that all fuzzy sets have strictly positive measure. In fact, if $P(A_i) > 0$ $1 \leqslant i \leqslant r$, $P(A_i) = 0$ $r+1 \leqslant i \leqslant l$, $P(B_j) > 0$ $1 \leqslant j \leqslant s$, and $P(B_j) = 0$ $s+1 \leqslant j \leqslant m$, we put $\xi' =$

$\{A_1,\ldots, A_{r-1}, A_r + \cdots + A_l\}$ and $\eta' = \{B_1,\ldots, B_{s-1}, B_s + \cdots + B_m\}$, where $A + B$ of fuzzy sets A and B means $m_{A+B} = m_A + m_B$. Then $H(\xi|\eta) = H(\xi'|\eta')$.

(i)
$$\begin{aligned}
H(\xi\eta|\zeta) &= -\sum_{i,j,k} P(A_iB_jC_k)\log\frac{P(A_iB_jC_k)}{P(C_k)} \\
&= -\sum_{i,j,k} P(A_iB_jC_k)\log\frac{P(A_iC_k)}{P(C_k)} \\
&\quad -\sum_{i,j,k} P(A_iB_jC_k)\log\frac{P(A_iB_jC_k)}{P(A_iC_k)} \\
&= -\sum_{i,k} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)} \\
&\quad -\sum_{i,j,k} P(A_iB_jC_k)\log\frac{P(A_iB_jC_k)}{P(A_iC_k)} \\
&= H(\xi|\zeta) + H(\eta|\xi\zeta).
\end{aligned}$$

(ii) Put $\zeta = \{X\}$ in (i).

(iii) From the definition $\xi < \eta$, for each i there is a subset $J(i)$ of $\{1,\ldots, m\}$ such that $m_{A_i} = \sum_{j\in J(i)} m_{B_j}$, $\bigcup_{i=1}^{l} J(i) = \{1,\ldots, m\}$, and $J(i) \cap J(h) = \varnothing$ $(i \neq h)$.

$$\begin{aligned}
H(\xi|\zeta) &= -\sum_{i,k} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)} \\
&= -\sum_{i,k}\left(\sum_{j\in J(i)} P(B_jC_k)\right)\log\frac{P(A_iC_k)}{P(C_k)} \\
&\leqslant -\sum_{i,j} P(B_jC_k)\log\frac{P(B_jC_k)}{P(C_k)} = H(\eta|\zeta).
\end{aligned}$$

(iv) Put $\zeta = \{X\}$ in (iii).

(v) We shall use the fact that the function $\phi(t) = -t\log t$ is concave. From $\eta < \zeta$, for each j there is a subset $K(j)$ of $\{1,\ldots, n\}$ such that $m_{B_j} = \sum_{k\in K(j)} m_{C_k}$. Since $P(B_j) = \sum_{k\in K(j)} P(C_k)$ and $\phi(t)$ is concave,

$$\begin{aligned}
&-\sum_{k\in K(j)} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)} \\
&\qquad = -P(B_j)\sum_{k\in K(j)}\frac{P(C_k)}{P(B_j)}\frac{P(A_iC_k)}{P(C_k)}\log\frac{P(A_iC_k)}{P(C_k)}
\end{aligned}$$

$$\leqslant -P(B_j)\left(\sum_{k\in K(j)} \frac{P(C_k)}{P(B_j)}\frac{P(A_iC_k)}{P(C_k)}\right)$$

$$\times \log\left(\sum_{k\in K(j)} \frac{P(C_k)}{P(B_j)}\frac{P(A_iC_k)}{P(C_k)}\right)$$

$$= -P(B_j)\frac{P(A_iB_j)}{P(B_j)}\log\frac{P(A_iB_j)}{P(B_j)}$$

$$= -P(A_iB_j)\log\frac{P(A_iB_j)}{P(B_j)}.$$

So we get

$$-\sum_{j=1}^{m} P(A_iB_j)\log\frac{P(A_iB_j)}{P(B_j)} \geqslant -\sum_{j=1}^{m}\sum_{k\in K(j)} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)}$$

$$= -\sum_{k=1}^{n} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)}$$

and therefore

$$H(\xi|\eta) = -\sum_{i,j} P(A_iB_j)\log\frac{P(A_iB_j)}{P(B_j)}$$

$$\geqslant -\sum_{i,k} P(A_iC_k)\log\frac{P(A_iC_k)}{P(C_k)} = H(\xi|\eta).$$

(vi) Put $\eta = \{X\}$ in (v).

(vii) By (i) and (v), $H(\xi\eta|\zeta) = H(\xi|\zeta) + H(\eta|\xi\zeta) \leqslant H(\xi|\zeta) + H(\eta|\zeta)$.

(viii) Put $\zeta = \{X\}$ in (vii).

This completes the proof.

DEFINITION 10. Let ξ and η be finite partitions of fuzzy sets. The information $I(\xi:\eta)$ about ξ obtained by η is defined as follows:

$$I(\xi:\eta) = H(\xi) - H(\xi|\eta).$$

PROPOSITION 11. *$I(\xi:\eta) \geqslant 0$ and $I(\xi:\eta) = I(\eta:\xi)$.*

Proof. By Theorem 9 (vi), $H(\xi) - H(\xi|\eta) \geqslant 0$.

By Theorem 9 (ii), $I(\xi:\eta) = H(\xi) - H(\xi|\eta) = H(\xi) + H(\eta) - H(\xi\eta) = I(\eta:\xi)$.

DEFINITION 12. Finite partitions $\xi = \{A_1,..., A_l\}$ and $\eta = \{B_1,..., B_m\}$ of fuzzy sets are said to be independent if $P(A_iB_j) = P(A_i)\,P(B_j)$ for all i, j.

PROPOSITION 13. *If finite partitions ξ and η are independent, $I(\xi:\eta)=0$.*

Proof. Since

$$H(\xi|\eta) = -\sum_{i,j} P(A_iB_j)\log\frac{P(A_iB_j)}{P(B_j)}$$

$$= -\sum_{i,j} P(A_i)\,P(B_j)\log\frac{P(A_i)\,P(B_j)}{P(B_j)} = H(\xi),$$

$$I(\xi:\eta) = H(\xi) - H(\xi|\eta) = 0.$$

REFERENCES

1. N. ABRAMSON, "Information Theory and Coding," McGraw–Hill, New York, 1963.
2. T. OKUDA, H. TANAKA, AND K. ASAI, A formulation of fuzzy decision problems with fuzzy information using probability measures of fuzzy events, *Inform. Control* **38** (1978), 135–147.
3. P. WALTERS, "Ergodic Theory-Introductory Lectures," Lecture Notes in Mathematics No. 458, Springer-Verlag Berlin, 1975.
4. L. A. ZADEH, Fuzzy sets, *Inform. Control* **8** (1965), 338–353.
5. L. A. ZADEH, Probability measures of fuzzy events, *J. Math. Anal. Appl.* 23 (1968), 421–427.